Master's thesis in Mathematics

Associating abelian varieties to weight-2 modular forms: the Eichler-Shimura construction

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Introduction

The famous **Taniyama-Shimura-Weil conjecture**, which recently became the **Modularity Theorem** after the work of Wiles, Breuil, Conrad, Diamond and Taylor, can be stated as follows:

Theorem

Any elliptic curve E defined over $\mathbb{Q}$ with conductor N is **modular**: there exists a modular form $f \in S_2(\Gamma_0(N))$ such that

$$L(s,E) = L(s,f).$$

Equivalently, $a_p(E) = a_p(f)$ for all primes p.

This arose as a converse of the following construction of Eichler and Shimura:

Theorem *(Eichler-Shimura, Carayol, Langlands, Deligne)*

Let $f \in S_2(\Gamma_0(N))$ be a newform. There exists an abelian variety A_f such that

1. A_f is defined over $\mathbb{Q}$.
2. A_f has dimension $[K_f : \mathbb{Q}]$, where $K_f = \mathbb{Q}(\{a_n(f) : n \geq 0\})$. In particular, if the Fourier coefficients of f are all rationals, then A_f is an elliptic curve.
3. A_f and f are related by their L-functions: we have
$$L(A_f, s) = \prod_\tau L(f_\tau, s),$$
where the product is over the complex embeddings $\tau : K_f \to \mathbb{C}$. Alternatively, $a_p(A_f) = \sum_\tau a_p(f_\tau)$ for all primes p.

For example, we see that the unique newform in $\Gamma_0(11)$, whose Fourier expansion has the first terms

$$q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + \ldots$$

corresponds to the elliptic curve

$$y^2 + y = x^3 - x^2 - 10x - 20. \qquad (1)$$

1 What is...

An abelian variety?

An abelian variety is a group in the category of projective abelian varieties. This generalizes *elliptic curves*, projective curves given by equations like (**??**), which admit a commutative group law given by morphisms.

A compact Riemann surface?

A Riemann surface is a complex analytic manifold of dimension 1. An extremely interesting property of *compact* Riemann surfaces is that they are projective algebraic curves.

A Jacobian variety?

If X is a compact Riemann surface, we define its Jacobian by

$$\mathrm{Jac}(X) = \Omega^1_{\mathrm{hol}}(X)^*/\Lambda,$$

where Λ corresponds to functionals given by integration around loops. Abel and Jacobi proved that $\mathrm{Jac}(X) \cong \mathrm{Pic}^0(X)$ as groups. By Riemann-Roch and Abel-Jacobi, $\mathrm{Jac}(X)$ is a complex torus. We can moreover prove that the latter admits a compatible structure (as Lie group) of *complex abelian variety*.

A modular form?

Recall that there is a left-action of $\mathrm{SL}_2(\mathbb{Z})$ on $\hat{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ given by *fractional linear transformations*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}.$$

A *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup given by congruence relations modulo a fixed integer. For example,

$$\Gamma_0(N) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N\right\}.$$

A weight-k **meromorphic modular form** with respect to a congruence subgroup Γ is a meromorphic function on $\hat{\mathbb{H}}$ (which has a suitable Riemann surface structure) such that

$$f(\gamma z) = (cz+d)^k f(z) \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,\ z \in \mathbb{H}.$$

The complex space of such objects is denoted $A_k(\Gamma)$. If f is holomorphic, we say that it is a **modular form** and we write $f \in M_k(\Gamma)$. In this case, we can show that f has a *Fourier expansion*

$$f(z) = \sum_{n\geq 0} a_n(f) q^n,$$

where $a_n(f) \in \mathbb{C}$. If $a_0(f) = 0$, we say that f is a **cusp form**, and we write $f \in S_k(\Gamma)$.

A L-function ?

An important idea in number theory is to attach meromorphic functions to various objects, encoding deep arithmetic properties of the latter. These functions will often have established or conjectured continuations to $\mathbb{C}$. In particular, there are L-functions attached to modular forms and to abelian varieties.

Modular forms

If $f = \sum_{n\geq 0} a_n(f) q^n \in M_k(\Gamma_0(N))$, we let $L(f,s) = \sum_{n\geq 0} \frac{a_n(f)}{n^s}$ be the *L-series* of f.

Abelian varieties

Let A be an abelian variety defined over $\mathbb{Q}$. For any prime p, we can consider a reduction A_p of A modulo p. For all but finitely many primes, this is an abelian variety over $\mathbb{F}_p$ and the Frobenius morphism $\sigma_p : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ induces a morphism $\sigma_p : A_p \to A_p$. The *L-function associated to A* is defined as

$$L(A,s) = \prod_p L_p(A,s),$$

where the local factor $L_p(A,s)$ at a place of good reduction is defined using the Frobenius morphism σ_p.

2 Idea of the construction

The idea of the Eichler-Shimura construction is the following:

1. Consider the **Jacobian varieties of modular curves**, which are compact Riemann surfaces strongly related to modular forms.
2. The **Hecke algebra** acts compatibly on modular forms, modular curves, and the Jacobians of the latter.
3. By **quotienting** the Jacobian of a modular curve by the action of a well-chosen subgroup of the Hecke algebra, we obtain the desired abelian variety.

3 Modular curves

If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we define the modular curves

$$Y(\Gamma) = \Gamma\backslash\mathbb{H} \subset X(\Gamma) = \Gamma\backslash\hat{\mathbb{H}},$$

and we prove that $X(\Gamma)$ has the structure of a **compact Riemann surface** (and hence it is also a **projective algebraic curve**). The link with modular forms is the following, coming from the fact that meromorphic modular forms are Γ-invariant meromorphic objects, and so are meromorphic forms on $X(\Gamma)$.

Proposition

For any even integer k, there is an isomorphism of complex vector spaces

$$A_k(\Gamma) \cong \Omega^{k/2}(X(\Gamma)),$$

which (co)restricts to an isomorphism $S_2(\Gamma) \cong \Omega^1_{\mathrm{hol}}(X(\Gamma))$.

4 The Hecke operators

For any integer $n \geq 0$, we have **Hecke operators** T_n and $\langle n\rangle$ acting compatibly on:

- Modular forms and cusp forms.
- Divisor groups of modular curves.
- Jacobian varieties (and thus Picard groups) of modular curves.

The commutative algebra $T_\mathbb{Z}$ generated by the Hecke operators is called the *Hecke algebra.*

5 Eigenforms and newforms

If we define a subspace $S_k(\Gamma_0(N))^{\mathrm{new}} \subset S_k(\Gamma_0(N))$ of cusp forms which "do not arise from lower levels N", we see $S_k(\Gamma_0(N))^{\mathrm{new}}$ admits a basis of forms which are *simultaneous eigenvectors for all Hecke operators*, called **eigenforms**. If $f \in S_k(\Gamma_0(N))$ is a normalized (i.e. $a_1(f) = 1$) eigenform, we say that f is a **newform** and in this case, we have

$$T_n f = a_n(f) f \text{ for all } n \geq 0.$$

Moreover, the L-function of f can then be written as

$$L(f,s) = \prod_p L_p(f,s),$$

where the *local factor* $L_p(f,s)$ is $(1 - a_p(f)p^{-s} + p^{k-1-2s})^{-1}$.

6 Definition of A_f

If $f \in S_2(\Gamma_0(N))$ is a newform, we have by definition a morphism

$$\lambda_f : T_\mathbb{Z} \to \mathbb{C}$$

such that $Tf = \lambda_f(T)f$ for all $T \in T_\mathbb{Z}$. We define the **abelian variety associated to** f by

$$A_f = \frac{\mathrm{Jac}(X_0(N))}{(\ker \lambda_f)\,\mathrm{Jac}(X_0(N))},$$

where $X_0(N) = X(\Gamma_0(N))$. Since the Hecke algebra acts by morphisms on $\mathrm{Jac}(X_0(N))$, we have that $(\ker \lambda_f)\,\mathrm{Jac}(X_0(N))$ is an abelian subvariety of $\mathrm{Jac}(X_0(N))$. Thus, we can show that A_f is indeed a complex abelian variety itself.

Proposition

The dimension of A_f (as an abelian variety or as a complex torus) is equal to $[K_f : \mathbb{Q}]$.

Note that $T_\mathbb{Z}/\ker\lambda_f \cong \mathbb{Z}[\{a_n(f)\}]$ acts on A_f by morphisms.

7 Definition over $\mathbb{Q}$

By analyzing its function field, we see that the modular curve $X_0(N)$, as an algebraic curve, can actually be defined over $\mathbb{Q}$. Moreover, the same holds true for its Jacobian and the Hecke operators, leading to:

Proposition

The complex abelian variety A_f can be defined over $\mathbb{Q}$.

8 The Eichler-Shimura relation

Let us then consider the modular curve $X_0(N)$ as an algebraic curve defined over $\mathbb{Q}$. For all but finitely many primes p, the reduction $X_0(N)_p$ is also an algebraic curve and the morphism T_p on $\mathrm{Pic}^0(X_0(N))$ reduce to a morphism on $\mathrm{Pic}^0(X_0(N)_p)$. The Eichler-Shimura relation computes the latter in terms of the Frobenius morphism:

Proposition *(Eichler-Shimura relation)*

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0(X_0(N)) & \xrightarrow{T_p} & \mathrm{Pic}^0(X_0(N)) \\ \downarrow & & \downarrow \\ \mathrm{Pic}^0(X_0(N)_p) & \xrightarrow{(\sigma_p)_* + (\sigma_p)^*} & \mathrm{Pic}^0(X_0(N)_p). \end{array}$$

The proof of this very interesting relationship is done as follows: the modular curve $Y_0(N) \subset X_0(N)$ corresponds to a moduli space of *enhanced elliptic curves*. There exists a model of $Y_0(N)$ such that reduction modulo p is compatible with "reducing the moduli space". We can then prove an analogue of the relation in the moduli space using the arithmetic of elliptic curves and transfer it back to the modular curve.

9 Equality of L-functions

Let $f \in S_2(\Gamma_0(N))$ be a newform. We have now two L-functions:

1. The **L-function of** f, whose local factor is
$$(1 - a_p(f)p^{-s} + p^{1-2s})^{-1}.$$
2. The **L-function of** A_f, whose local factor depends on the Frobenius σ_p.

To prove that they agree, we transfer the Eichler-Shimura relation to A_f:

Proposition

The diagram

$$\begin{array}{ccc} A_f & \xrightarrow{T_p = a_p(f)} & A_f \\ \downarrow & & \downarrow \\ (A_f)_p & \xrightarrow{\sigma_p + \hat{\sigma}_p} & (A_f)_p \end{array}$$

commutes, where $\hat{\sigma}_p$ is the dual isogeny.

- In the **case of elliptic curves** (i.e. $\dim A_f = 1$), we have
$$L_p(A_f, s) = (1 - a_p(A_f)p^{-s} + p^{1-2s})^{-1}$$
if p is a place of good reduction, so that it suffices to prove that $a_p(f) = a_p(A_f)$ to show that the local factors agree. But $\sigma_p + \hat{\sigma}_p = a_p(A_f)$, whence the above diagram gives that multiplication by $a_p(f) - a_p(A_f)$ does not surject on A_f, which implies that $a_p(f) = a_p(A_f)$ as integers.
- In the **general case**, we use the above diagram to connect:
  - The action of the Frobenius morphism on $(A_f)_p$, which is related to the L-function of A_f.
  - The action of T_p on A_f, which is related to $a_p(f)$, so to the local factor of the L-function of f.

References

Important references about this subject are the survey article Modular forms and modular curves *by Fred Diamond and John Im (1995), the book* Introduction to the Arithmetic Theory of Automorphic Functions *by Goro Shimura (1971) and the book* A First Course in Modular Forms *by Fred Diamond and Jerry Shurman (2006). A complete list of references along with this master's thesis can be found at* `http://sma.epfl.ch/~cperret`.